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SU(1,1) Lie Algebra Applied to the General Time-dependent Quadratic Hamiltonian System

J. R. Choi[1,2] and I. H. Nahm[1]

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Exact quantum states of the time-dependent quadratic Hamiltonian system are investigated using SU(1,1) Lie algebra. We realized SU(1,1) Lie algebra by defining appropriate SU(1,1) generators and derived exact wave functions using this algebra for the system. Raising and lowering operators of SU(1,1) Lie algebra expressed by multiplying a time-constant magnitude and a time-dependent phase factor. Two kinds of the SU(1,1) coherent states, i.e., even and odd coherent states and Perelomov coherent states are studied. We applied our result to the Caldirola–Kanai oscillator. The probability density of these coherent states for the Caldirola–Kanai oscillator converged to the center as time goes by, due to the damping constant γ. All the coherent state probability densities for the driven system are somewhat deformed.

KEY WORDS: SU(1,1) Lie algebra; time-dependent quadratic Hamiltonian system; coherent states.

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1. INTRODUCTION

A large number of solvable systems including time-dependent quadratic Hamiltonian systems (TDQHSs) which can be applied to various quantum systems may be classified to the type of SU(1,1) (Inomata *et al.*, 1992). The time-dependent quadratic Hamiltonian systems have been attracted interest in the literature (Lewis and Riesenfeld, 1969; Yeon *et al.*, 1997; Choi and Gweon, 2002) from the invention of the invariant operator method by Lewis (1967). Yeon *et al.*, investigated TDQHSs, which are connected by canonical transformations using invariant operator method (Yeon *et al.*, 1997). The TDQHSs have plentiful applications in various fields of physics such as quantum optics (Choi, 2003) mesoscopic electric circuit (Zhang *et al.*, 2001), and acoustics (Choi, 2004a). Dodonov applied TDQHSs to the derivation of the coherent states for a charged particle in a time-

[1] Department of Physics and Advanced Materials Science, Sun Moon University, Asan 336-708, Republic of Korea.
[2] e-mail: choiardor@hanmail.net.

dependent uniform perpendicular electric and magnetic fields (Dodonov *et al.*, 1972) and to the loss energy states of nonstationary quantum harmonic oscillator with a damping term (Dodonov and Man'ko, 1978). One of the typical types of the TDQHS is Caldirola–Kanai Hamiltonian system with a driving force. We applied Caldirola–Kanai Hamiltonian system to the description of the electromagnetic waves propagating through homogeneous conducting linear media in view of quantum mechanics (Choi, 2003) and the description of the quantum sound wave in a cylindrical conduit (Choi, 2004a).

Many kinds of phenomena such as quantum correlation, phase coherence, and squeezing effect in quantum systems may be explained in terms of the SU(1,1) Lie algebra and the generalized coherent states associated with these Lie algebras (Wódkiewicz and Eberly, 1985). The phase operators for the SU(1,1) Lie algebras, which plays an important role in describing nonclassical properties of light, have been investigated theoretically (Gerry, 1988). Beam splitters (Campos *et al.*, 1989) and quantum mechanical interferometers in quantum optics have been also analyzed by the SU(1,1) Lie algebra. The one mode bosonic realization of SU(1,1) Lie algebra can be used to describe the degenerate parametric amplifier.

The concept of coherent state, which was introduced by (Glauber, 1963) has attained an important position in quantum optics since the coherent states not only construct a very useful representation but also have physical substance. There exist two kinds of the SU(1,1) coherent states, i.e., even and odd coherent states (Dodonov *et al.*, 1974) and Perelomov coherent states (Perelomov, 1972). Perelomov coherent states is a special case of the two-photon coherent states of (Yuen, 1976) and investigated in connection with squeezed states of a single-mode field (Wódkiewicz and Eberly, 1985). The purpose of this paper is to investigate SU(1,1) Lie algebra for the TDQHS and to explain various quantum-mechanical properties of the system using this algebra. In the following section, we realize the SU(1,1) Lie algebra by introducing generators. In Section 3, SU(1,1) coherent states for the TDQHS are investigated. We applied our study to the Caldiroa–Kanai Hamiltonian system, which is an example of the TDQHS in Section 4. In the last section, we summarize previous sections with comparative explanation for the TDQHS and their coherent states.

2. ONE MODE TIME-DEPENDENT QUADRATIC HAMILTONIAN SYSTEM

In this section, we consider one mode TDQHS whose Hamiltonian is given by

$$\hat{H}(\hat{x}, \hat{p}, t) = \frac{1}{2}[A(t)\hat{p}^2 + B(t)(\hat{x}\hat{p} + \hat{p}\hat{x}) + C(t)\hat{x}^2] + D(t)\hat{x} + E(t)\hat{p} + F(t), \tag{1}$$

where $\hat{x}$ and $\hat{p}$ are canonical variables which satisfying $[\hat{x}, \hat{p}] = i\hbar$ and $A(t)-F(t)$ are real time-dependent functions that are differentiable with respect to time. In view of classical mechanics, we may use Hamiltonian dynamics

$$\dot{x} = \frac{\partial H(x,p,t)}{\partial p}, \qquad \dot{p} = -\frac{\partial H(x,p,t)}{\partial x}, \tag{2}$$

in order to derive the classical equation of motion for the system:

$$\frac{d^2x(t)}{dt^2} - \frac{\dot{A}}{A}\frac{dx(t)}{dt} + \left(AC + \frac{\dot{A}B}{A} - B^2 - \dot{B}\right)x(t) = -\frac{\dot{A}E}{A} + BE - AD + \dot{E}. \tag{3}$$

The terms in the right hand side of the above equation are related to the force exerted to the system. If we know a solution of coordinate satisfying the above equation, say it $x_0(t)$, we can also obtain corresponding momentum solution $p_0(t)$ from

$$p_0(t) = \frac{1}{A}\left[\frac{dx_0(t)}{dt} - Bx_0(t) - E\right]. \tag{4}$$

Classical solutions $x_0(t)$ and $p_0(t)$ consists of two parts, complementary functions $x_c(t)$ and $p_c(t)$ plus particular solutions $x_p(t)$ and $p_p(t)$ (Marion, 1970):

$$x_0(t) = x_c(t) + x_p(t), \tag{5}$$

$$p_0(t) = p_c(t) + p_p(t). \tag{6}$$

Many researchers have used SU(1,1) Lie algebra in order to facilitate the investigation of diverse phenomena in quantum systems. The Lie algebra of SU(1,1) in one mode consists of operators $\hat{K}_0$, $\hat{K}_1$, and $\hat{K}_2$. We define these operators as

$$\hat{K}_0 = \frac{1}{4\hbar\Omega}\left\{\Omega^2\frac{1}{s^2(t)}[\hat{x}-x_p(t)]^2 + \left[\frac{1}{A}\left(Bs(t)-\dot{s}(t)\right)[\hat{x}-x_p(t)] + s(t)[\hat{p}-p_p(t)]\right]^2\right\}, \tag{7}$$

$$\hat{K}_1 = \frac{1}{4\hbar\Omega}\left\{\Omega^2\frac{1}{s^2(t)}[\hat{x}-x_p(t)]^2 - \left[\frac{1}{A}\left(Bs(t)-\dot{s}(t)\right)[\hat{x}-x_p(t)] + s(t)[\hat{p}-p_p(t)]\right]^2\right\}, \tag{8}$$

$$\hat{K}_2 = -\frac{1}{2\hbar}\left\{\frac{1}{s(t)A}(Bs(t)-\dot{s}(t))[\hat{x}-x_p(t)]^2 + \frac{1}{2}\{[\hat{x}-x_p(t)][\hat{p}-p_p(t)] + [\hat{p}-p_p(t)][\hat{x}-x_p(t)]\}\right\}, \tag{9}$$

where Ω is some real positive constant and $s(t)$ is some time-dependent variable that satisfies the following differential equation

$$\ddot{s}(t) - \frac{\dot{A}}{A}\dot{s}(t) + \left(AC + \frac{\dot{A}B}{A} - B^2 - \dot{B}\right)s(t) - A^2\Omega^2\frac{1}{s^3(t)} = 0. \quad (10)$$

The mutual commutation relations between $\hat{K}_0$, $\hat{K}_1$, and $\hat{K}_2$ are

$$[\hat{K}_0,\ \hat{K}_1] = i\hat{K}_2, \qquad [\hat{K}_0,\ \hat{K}_2] = -i\hat{K}_1, \qquad [\hat{K}_1,\ \hat{K}_2] = -i\hat{K}_0. \quad (11)$$

The third equation in (11) is the characteristic of SU(1,1) which is different from SU(2). We can easily check that $\hat{K}_0$ is time-constant by differentiating it with respect to time:

$$\frac{d\hat{K}_0}{dt} = 0. \quad (12)$$

For this one mode TDQHSs, we define raising and lowering operators as

$$\hat{K}_+ = \hat{K}_1 + i\hat{K}_2, \qquad \hat{K}_- = \hat{K}_1 - i\hat{K}_2. \quad (13)$$

Note that a direct differentiation of the above two operators with respect to time results

$$\frac{d\hat{K}_+}{dt} = \frac{2i\Omega A(t)}{s^2(t)}\hat{K}_+, \quad (14)$$

$$\frac{d\hat{K}_-}{dt} = -\frac{2i\Omega A(t)}{s^2(t)}\hat{K}_-, \quad (15)$$

whose solutions are given by

$$\hat{K}_+(t) = \hat{K}_+(0)\exp\left(2i\int_0^t \frac{\Omega A(t')}{s^2(t')}dt'\right), \quad (16)$$

$$\hat{K}_-(t) = \hat{K}_-(0)\exp\left(-2i\int_0^t \frac{\Omega A(t')}{s^2(t')}dt'\right). \quad (17)$$

Thus, the magnitude of $\hat{K}_+$ and $\hat{K}_-$ are evidently constant with time. Additional commutation relations between SU(1,1) generators are

$$[\hat{K}_-,\ \hat{K}_+] = 2\hat{K}_0, \qquad [\hat{K}_0,\ \hat{K}_\pm] = \pm\hat{K}_\pm. \quad (18)$$

Since the Casimir operator can be calculated as

$$\hat{C} = \hat{K}_0^2 - \frac{1}{2}(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+) = -\frac{3}{16}, \quad (19)$$

the eigenvalue of $\hat{C}$ is $k(k-1) = -3/16$. Therefore, the Bargmann index is $k = 1/4$ or $k = 3/4$. For $k = 1/4$, the basis for the unitary space is a set of even boson number while $k = 3/4$, the basis for the unitary space is a set of odd boson number.

Let us denote the nth eigenvalues and eigenstates of $\hat{K}_0$ as λ_n and $|n\rangle$, respectively:

$$\hat{K}_0|n\rangle = \lambda_n|n\rangle. \tag{20}$$

Then, the actions of the SU(1,1) generators on $|n\rangle$ is

$$\hat{K}_+|n\rangle = \frac{1}{2}\sqrt{(n+1)(n+2)}|n+2\rangle, \tag{21}$$

$$\hat{K}_-|n\rangle = \frac{1}{2}\sqrt{n(n-1)}|n-2\rangle, \tag{22}$$

$$\hat{K}_0|n\rangle = \frac{1}{2}\left(n+\frac{1}{2}\right)|n\rangle, \tag{23}$$

$$\hat{K}_+\hat{K}_-|n\rangle = \frac{1}{4}n(n-1)|n\rangle. \tag{24}$$

The generators $\hat{K}_\pm$ can also be used to obtain the eigenstates in position space. Let us start with the ground and the first excited states defined by

$$\hat{K}_-|0\rangle = 0, \qquad \hat{K}_-|1\rangle = 0. \tag{25}$$

We see that the normalized solutions to the above equations in position space are

$$\langle x|0\rangle = \left(\frac{\Omega}{\pi s^2\hbar}\right)^{1/4} e^{ip_p(t)x/\hbar} \exp\left\{-\frac{1}{2s\hbar}\left[\Omega\frac{1}{s}+\frac{i}{A}(Bs-\dot{s})\right][x-x_p(t)]^2\right\}, \tag{26}$$

$$\langle x|1\rangle = \left(\frac{\Omega}{\pi s^2\hbar}\right)^{1/4} \frac{1}{\sqrt{2}} H_1\left(\sqrt{\frac{\Omega}{s^2\hbar}}[x-x_p(t)]\right) e^{ip_p(t)x/\hbar}$$
$$\times \exp\left\{-\frac{1}{2s\hbar}\left[\Omega\frac{1}{s}+\frac{i}{A}(Bs-\dot{s})\right][x-x_p(t)]^2\right\}, \tag{27}$$

where H_n is nth order Hermite polynomial. The higher order eigenstates can be derived by operating $\hat{K}_+$ appropriate times to Eqs. (26) and (27). So to speak, the even order ($n = 2l$) and the odd order ($n = 2l + 1$) higher eigenstates can be evaluated from

$$\langle x|2l\rangle = \frac{2^{2l}}{\sqrt{(2l)!}}(\hat{K}_+)^l\langle x|0\rangle, \tag{28}$$

$$\langle x|2l+1\rangle = \frac{2^{2l}}{\sqrt{(2l+1)!}}(\hat{K}_+)^l\langle x|1\rangle. \tag{29}$$

When we combine the results of the above two equations, the whole eigenstates can be written as

$$\langle x|n\rangle = \left(\frac{\Omega}{\pi s^2 \hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n \left(\sqrt{\frac{\Omega}{s^2 \hbar}}[x - x_p(t)]\right) e^{ip_p(t)x/\hbar}$$
$$\times \exp\left\{-\frac{1}{2s\hbar}\left[\Omega\frac{1}{s} + \frac{i}{A}(Bs - \dot{s})\right][x - x_p(t)]^2\right\}. \quad (30)$$

From Eq. (23), the eigenvalues of $\hat{K}_0$ are

$$\lambda_n = \frac{1}{2}\left(n + \frac{1}{2}\right). \quad (31)$$

The wave functions of the system, $\langle x|\psi_n\rangle$, are different from the eigenstates of $\hat{K}_0$, $\langle x|n\rangle$, by only some time-dependent phase factors $\exp[i\theta_n(t)]$ (Lewis and Riesenfeld, 1969):

$$\langle x|\psi_n\rangle = \langle x|n\rangle \exp[i\theta_n(t)]. \quad (32)$$

By substituting Eq. (32) into Schrödinger equation, we can find the phases $\theta_n(t)$ as

$$\theta_n(t) = -\left(n + \frac{1}{2}\right)\int_0^t \frac{A(t')\Omega}{s^2(t')}dt'$$
$$-\frac{1}{\hbar}\int_0^t \left[\frac{1}{2}\left[A(t')p_p^2(t') - C(t')x_p^2(t')\right] + E(t')p_p(t') + F(t')\right]dt'. \quad (33)$$

The wave functions Eq. (32) agree with those of (Choi, 2004b) which are obtained using another method.

3. SU(1,1) COHERENT STATES

The coherent states of usual harmonic oscillator can be generalized to that of the TDQHS. The even and odd coherent states (Dodonov *et al.*, 1974; Choi, 2004b) $|\alpha_+\rangle$ and $|\alpha_-\rangle$ are the eigenstates of $\hat{K}_-$:

$$\hat{K}_-|\alpha_\pm\rangle = \frac{1}{2}\alpha^2|\alpha_\pm\rangle. \quad (34)$$

It is well known that $|\alpha_+\rangle$ and $|\alpha_-\rangle$ can be expressed in terms of $|n\rangle$ as

$$|\alpha_+\rangle = \frac{1}{\sqrt{\cosh|\alpha|^2}}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{(2n)!}}|2n\rangle, \quad (35)$$

$$|\alpha_-\rangle = \frac{1}{\sqrt{\sinh|\alpha|^2}}\sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}}|2n+1\rangle. \quad (36)$$

This implies that the even and odd coherent states are symmetry and anti-symmetry combination of the Glauber coherent states, respectively. To obtain position representation of these coherent states, we multiply both sides of the above two equations by the position eigenbra $\langle x|$ on the left and perform a little algebra using Eqs. (28) and (29). Thus,

$$\langle x|\alpha_+\rangle = \left(\frac{\Omega}{s^2\hbar\pi}\right)^{1/4} \frac{1}{\sqrt{\cosh|\alpha|^2}} \cosh\left(\alpha\sqrt{\frac{2\Omega}{s^2\hbar}}[x-x_p(t)]\right) e^{ip_p(t)x/\hbar} \times \exp\left\{-\frac{1}{2s\hbar}\left[\frac{\Omega}{s}+\frac{i}{A}(Bs-\dot{s})\right][x-x_p(t)]^2-\frac{1}{2}\alpha^2\right\}, \quad (37)$$

$$\langle x|\alpha_-\rangle = \left(\frac{\Omega}{s^2\hbar\pi}\right)^{1/4} \frac{1}{\sqrt{\sinh|\alpha|^2}} \sinh\left(\alpha\sqrt{\frac{2\Omega}{s^2\hbar}}[x-x_p(t)]\right) e^{ip_p(t)x/\hbar} \times \exp\left\{-\frac{1}{2s\hbar}\left[\frac{\Omega}{s}+\frac{i}{A}(Bs-\dot{s})\right][x-x_p(t)]^2-\frac{1}{2}\alpha^2\right\}. \quad (38)$$

By considering the second part of Eq. (13) and Eq. (34), we can express α as

$$\alpha = \sqrt{\frac{1}{2\hbar\Omega}}\left\{\left[\frac{\Omega}{s}+i\frac{1}{A}(Bs-\dot{s})\right]x_c(t)+isp_c(t)\right\}. \quad (39)$$

Using the above equation, Eqs. (37) and (38) can be rewritten as

$$\langle x|\alpha_\pm\rangle = \left(\frac{\Omega}{4s^2\hbar\pi}\right)^{1/4} \exp\left(-i\frac{p_c(t)x_c(t)}{2\hbar}\right) e^{ip_p(t)x/\hbar} \times\left\{\exp\left[-\frac{1}{2s\hbar}\left(\frac{\Omega}{s}+i\frac{Bs-\dot{s}}{A}\right)(x-x_0(t))^2+i\frac{[x-x_p(t)]p_c(t)}{\hbar}\right] \pm\exp\left[-\frac{1}{2s\hbar}\left(\frac{\Omega}{s}+i\frac{Bs-\dot{s}}{A}\right)(x+x_0(t)-2x_p(t))^2 -i\frac{[x-x_p(t)]p_c(t)}{\hbar}\right]\right\}\left\{1\pm\exp\left\{-\frac{1}{\hbar\Omega}\left[\frac{\Omega^2}{s^2}x_c^2(t) +\left(\frac{1}{A}(Bs-\dot{s})x_c(t)+sp_c(t)\right)^2\right]\right\}\right\}^{-1/2}. \quad (40)$$

The expectation values of canonical variables and their square in these coherent states are

$$\langle\alpha_+|\hat{x}|\alpha_+\rangle = \langle\alpha_-|\hat{x}|\alpha_-\rangle = x_p(t), \quad (41)$$

$$\langle\alpha_+|\hat{p}|\alpha_+\rangle = \langle\alpha_-|\hat{p}|\alpha_-\rangle = p_p(t), \tag{42}$$

$$\langle\alpha_+|\hat{x}^2|\alpha_+\rangle = \frac{s^2\hbar}{2\Omega}(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 \tanh|\alpha|^2 + 1) + x_p^2(t), \tag{43}$$

$$\langle\alpha_-|\hat{x}^2|\alpha_-\rangle = \frac{s^2\hbar}{2\Omega}(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 \coth|\alpha|^2 + 1) + x_p^2(t), \tag{44}$$

$$\begin{aligned}\langle\alpha_+|\hat{p}^2|\alpha_+\rangle = -\frac{\hbar\Omega}{2s^2}\Bigg\{ & \left[1 - i\frac{s(Bs-\dot{s})}{A\Omega}\right]^2\alpha^2 + \left[1 + i\frac{s(Bs-\dot{s})}{A\Omega}\right]^2\alpha^{*2} \\ & - \left[1 + \left(\frac{s(Bs-\dot{s})}{A\Omega}\right)^2\right](2|\alpha|^2\tanh|\alpha|^2 + 1)\Bigg\} + p_p^2(t),\end{aligned} \tag{45}$$

$$\begin{aligned}\langle\alpha_-|\hat{p}^2|\alpha_-\rangle = -\frac{\hbar\Omega}{2s^2}\Bigg\{ & \left[1 - i\frac{s(Bs-\dot{s})}{A\Omega}\right]^2\alpha^2 + \left[1 + i\frac{s(Bs-\dot{s})}{A\Omega}\right]^2\alpha^{*2} \\ & - \left[1 + \left(\frac{s(Bs-\dot{s})}{A\Omega}\right)^2\right](2|\alpha|^2\coth|\alpha|^2 + 1)\Bigg\} + p_p^2(t).\end{aligned} \tag{46}$$

The density operator in these states can be defined as

$$\hat{\rho} = P_+|\alpha_+\rangle\langle\alpha_+| + P_-|\alpha_-\rangle\langle\alpha_-|, \tag{47}$$

where P_+ and P_- are the weights of the even and odd coherent states in the initial state. The probabilities of finding n' quanta in the coherent states are

$$P_{n'} = \langle n'|\hat{\rho}|n'\rangle. \tag{48}$$

The substitution of Eq. (47) into the above equation gives

$$P_{n'} = \frac{P_+}{\cosh|\alpha|^2}\frac{|\alpha|^{4n}}{(2n)!}\delta_{n',2n} + \frac{P_-}{\sinh|\alpha|^2}\frac{|\alpha|^{4n+2}}{(2n+1)!}\delta_{n',2n+1}. \tag{49}$$

The other SU(1,1) coherent state, so-called Perelomov coherent states, can be defined as (Perelomov, 1972; Gerry, 1991)

$$\begin{aligned}|\xi;k\rangle_{\mathrm{P}} &= \exp\left[\frac{1}{2}(\alpha^2\hat{K}_+ - \alpha^{*2}\hat{K}_-)\right]|0\rangle_k \\ &= (1-|\xi|^2)^k \sum_{n=0}^{\infty}\left(\frac{(n+2k-1)!}{n!(2k-1)!}\right)^{1/2}\xi^n|n\rangle_k,\end{aligned} \tag{50}$$

where

$$\xi = \frac{\alpha^2}{|\alpha|^2}\tanh(|\alpha|^2/2). \tag{51}$$

For $k = 1/4$ and $k = 3/4$, Eq. (50) becomes (Gerry, 1991)

$$|\xi; 1/4\rangle_{\mathrm{P}} = (1 - |\xi|^2)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \xi^n |2n\rangle, \tag{52}$$

$$|\xi; 3/4\rangle_{\mathrm{P}} = (1 - |\xi|^2)^{3/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!} \xi^n |2n+1\rangle. \tag{53}$$

We can express the density operator in this coherent state as

$$\hat{\rho} = P_+ |\xi; 1/4\rangle_{\mathrm{P}}\, {}_{\mathrm{P}}\langle\xi; 1/4| + P_- |\xi; 3/4\rangle_{\mathrm{P}}\, {}_{\mathrm{P}}\langle\xi; 3/4|. \tag{54}$$

Then, the probabilities of finding n' quanta in this state can be evaluated

$$P_{n'} = \frac{(1 - |\xi|^2)^{1/2}}{2^{2n}(n!)^2} |\xi|^{2n} [P_+(2n)!\delta_{n',2n} + P_-(1 - |\xi|^2)(2n+1)!\delta_{n',2n+1}]. \tag{55}$$

4. APPLICATION TO THE CALDIROLA–KANAI HAMILTONIAN SYSTEM

In order to apply one mode TDQHS to the real system, let us consider driven Caldirola–Kanai Hamiltonian system (Kanai, 1948). In this case, the functions in Eq. (1) are given by

$$A = \frac{1}{M} e^{-\gamma t}, \qquad C = M\omega_0^2 e^{\gamma t}, \qquad D = M e^{\gamma t} f(t), \tag{56}$$

and all other functions are zero, where M, γ and ω_0 are real positive constants. We choose driving force $f(t)$ as

$$f(t) = f_0 \cos(\omega_d t + \phi_d), \tag{57}$$

where amplitude f_0, frequency ω_d and arbitrary phase ϕ_d are real constants. Then, the Hamiltonian reduces to

$$\hat{H} = \frac{\hat{p}^2}{2M} e^{-\gamma t} + \frac{1}{2} M e^{\gamma t} \left[\omega_0^2 \hat{x}^2 - 2 f(t) \hat{x}\right]. \tag{58}$$

In this case, the classical equation of motion Eq. (3) becomes

$$\ddot{x}_0(t) + \gamma \dot{x}_0(t) + \omega_0^2 x_0(t) = f(t), \tag{59}$$

and, Eq. (10) can be rewritten as

$$\ddot{s}(t) + \gamma \dot{s}(t) + \omega_0^2 s(t) - \frac{\Omega^2}{M^2} e^{-2\gamma t} \frac{1}{s^3(t)} = 0. \tag{60}$$

We introduce a solution that satisfies Eq. (60) as (Pedrosa *et al.*, 1997)

$$s(t) = \sqrt{\frac{\Omega}{M}}\tilde{\omega}^{-1/2} \exp\left(-\frac{\gamma}{2}t\right), \tag{61}$$

where $\tilde{\omega}$ is modified frequency that given by

$$\tilde{\omega} = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}. \tag{62}$$

We can also derive complementary functions and particular solutions as

$$x_c(t) = x_{c0}e^{-\gamma t/2}\cos(\tilde{\omega}t + \phi), \tag{63}$$

$$p_c(t) = -Mx_{c0}e^{\gamma t/2}\left[\frac{\gamma}{2}\cos(\tilde{\omega}t + \phi) + \tilde{\omega}\sin(\tilde{\omega}t + \phi)\right], \tag{64}$$

$$x_p(t) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \gamma^2\omega_d^2}}\cos(\omega_d t + \phi_d - \delta), \tag{65}$$

$$p_p(t) = -\frac{Mf_0\omega_d}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \gamma^2\omega_d^2}}e^{\gamma t}\sin(\omega_d t + \phi_d - \delta), \tag{66}$$

where x_{c0} is amplitude of oscillation at $t = 0$, ϕ is initial phase and

$$\delta = \tan^{-1}\frac{\gamma\omega_d}{\omega_0^2 - \omega_d^2}. \tag{67}$$

If we use Eqs. (63) and (64), Eq. (39) reduces to

$$\alpha(t) = \alpha_0 e^{-i(\tilde{\omega}t+\phi)}, \tag{68}$$

for the Caldirola–Kanai oscillator where $\alpha_0 = \sqrt{M\tilde{\omega}/(2\hbar)}x_{c0}$. In terms of Eq. (61) and Eqs.(63)–(66), the system can be described explicitly. Figures 1 and 2 are the probability density in even and in odd coherent states for the Caldirola–Kanai oscillator. The even and the odd coherent states are important since they can be applied to describe the motion of a trapped ion (de Matos Filho and Vogel, 1996). Matos and Vogel suggested a scheme for preparing even and odd coherent states of a trapped ion based on laser excitation of two vibronic transitions (de Matos Filho and Vogel, 1996). We also depicted the probability density in Perelomov coherent states in Figs. 3 and 4. From all figures, we can confirm that the probability density in the coherent states for the Caldirola–Kanai oscillator converges to the center as time goes by, due to the damping constant γ.

5. SUMMARY

In this paper, SU(1,1) Lie algebric formulation that enabled us to investigate the quantum properties of the TDQHS are described. We realized SU(1,1) Lie

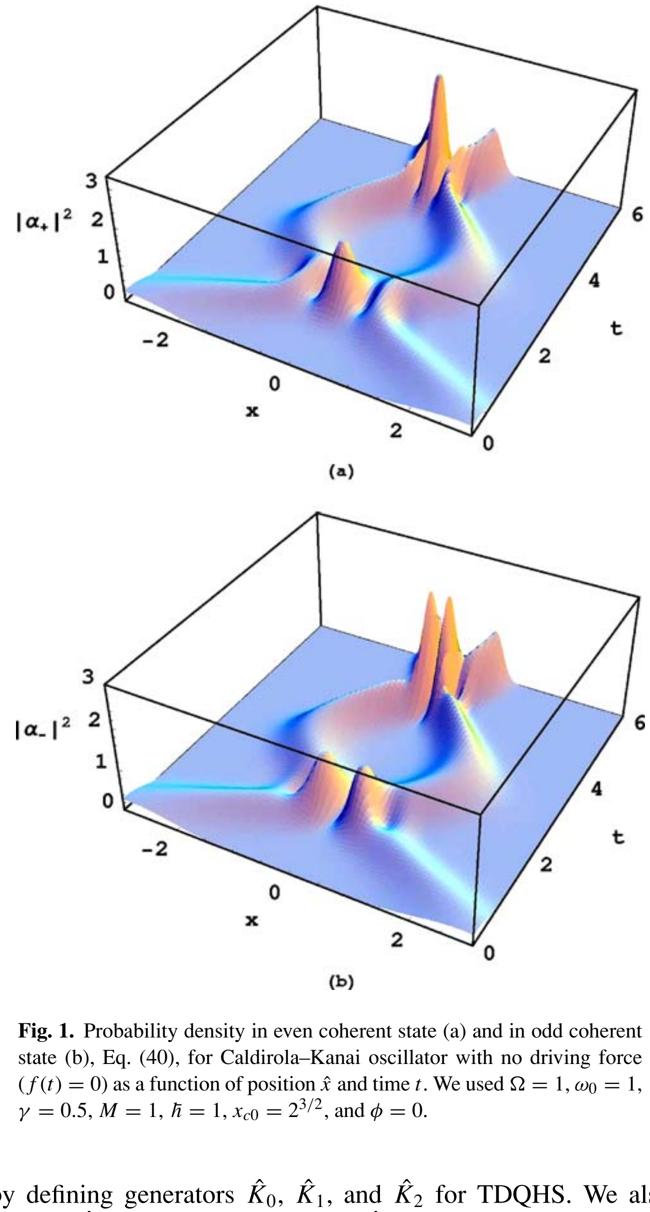


Fig. 1. Probability density in even coherent state (a) and in odd coherent state (b), Eq. (40), for Caldirola–Kanai oscillator with no driving force ($f(t) = 0$) as a function of position $\hat{x}$ and time t. We used $\Omega = 1$, $\omega_0 = 1$, $\gamma = 0.5$, $M = 1$, $\hbar = 1$, $x_{c0} = 2^{3/2}$, and $\phi = 0$.

algebra by defining generators $\hat{K}_0$, $\hat{K}_1$, and $\hat{K}_2$ for TDQHS. We also defined raising operator $\hat{K}_+$ and lowering operator $\hat{K}_-$ for the system. These raising and lowering operators of SU(1,1) Lie algebra can be expressed by multiplying a time-constant magnitude and a time-dependent phase factor [see Eqs. (16) and (17)]. We confirmed that the generators $\hat{K}_0$ is constant with time. Exact wave

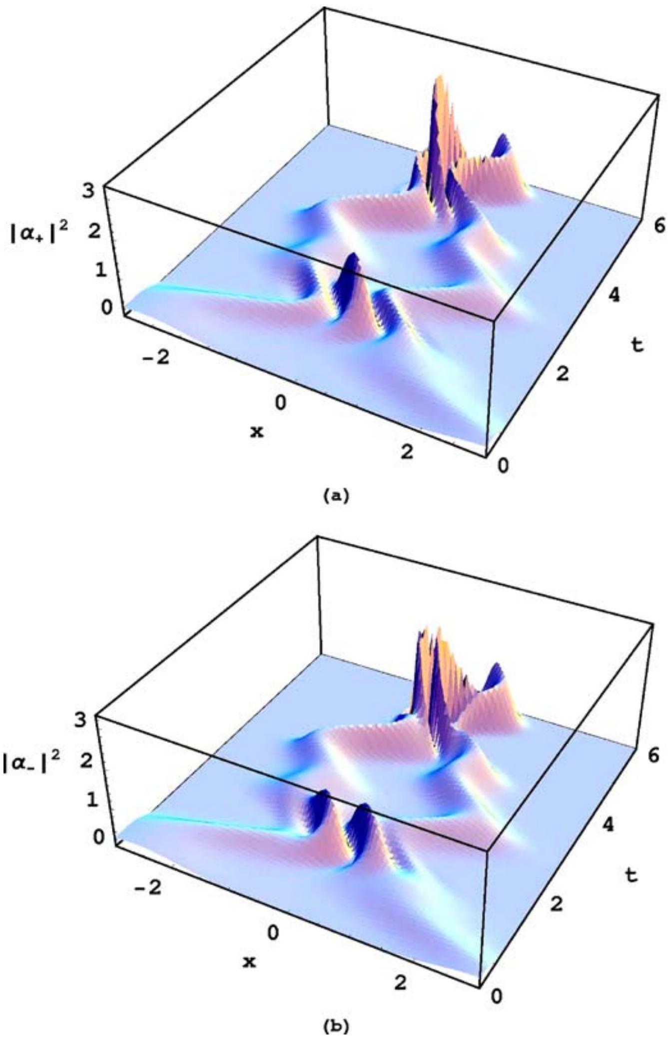


Fig. 2. Same as in Fig. 1, but with a driving force Eq. (57). We used $f_0 = 5$, $\omega_d = 4.5$, and $\phi_d = 0$.

functions for the system are derived using SU(1,1) Lie algebra. Two kind of the generalized SU(1,1) coherent states, i.e., even and odd coherent states and Perelomov coherent states are studied. Even and odd coherent states, Eq. (40), are represented in terms of the classical solutions for coordinate and momentum, i.e., complementary solutions $x_c(t)$ and $p_c(t)$ plus particular solutions $x_p(t)$ and $p_p(t)$.

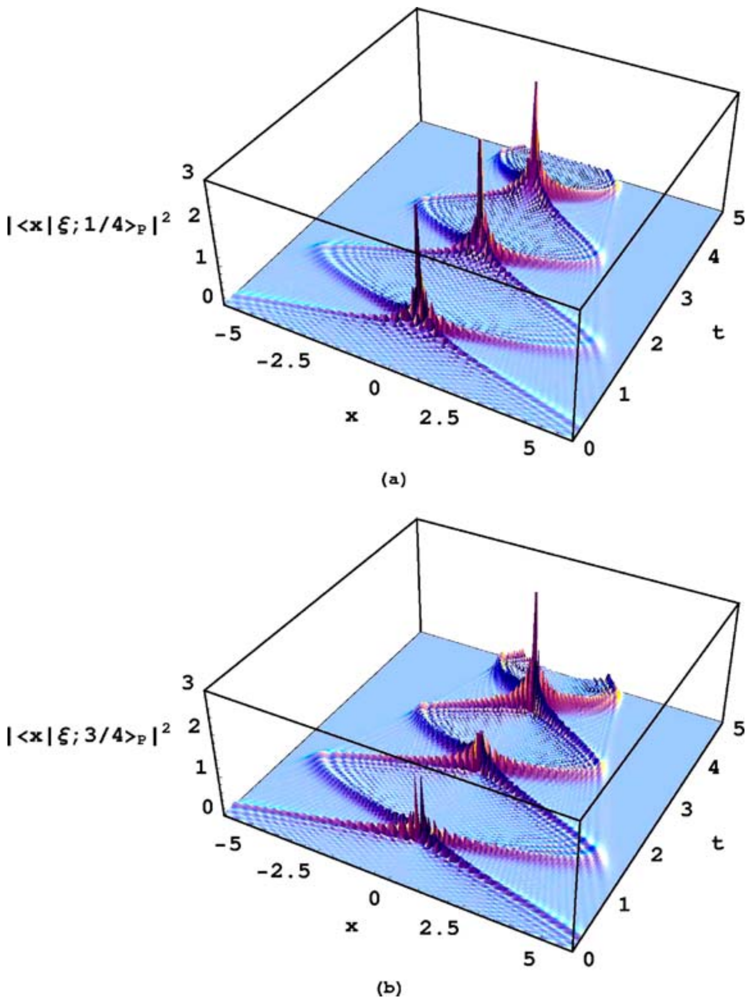
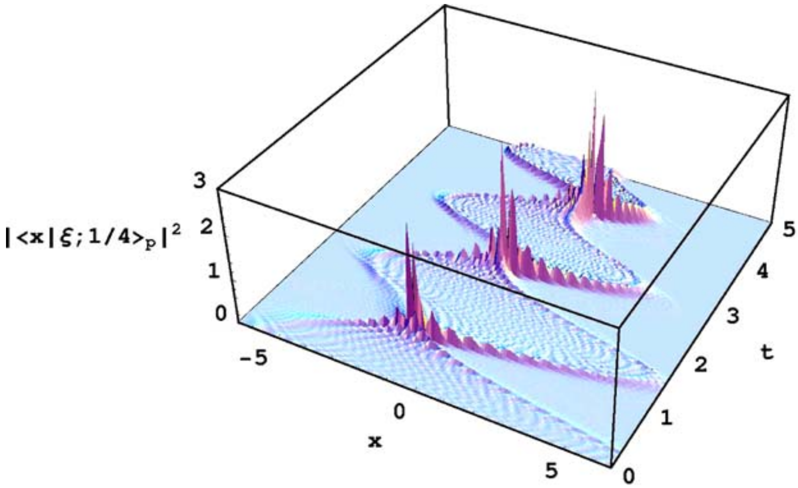


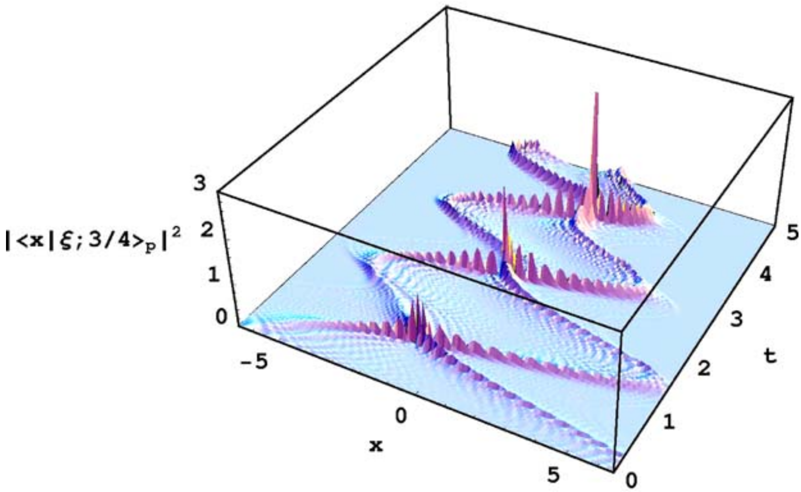
Fig. 3. Probability density in Perelomov coherent state, Eqs. (52) and (53) for Caldirola–Kanai oscillator with no driving force ($f(t) = 0$) as a function of position $\hat{x}$ and time t. Figure (a) is for $k = 1/4$, and figure (b) for $k = 3/4$. We used $\Omega = 1$, $\omega_0 = 2$, $\gamma = 0.5$, $M = 1$, $\hbar = 1$, $x_{c0} = 2^{3/2}$, and $\phi = 0$.

Expectation values for coordinate and momentum and their square in even and odd coherent states and in Perelomov coherent states are derived.

We applied our study to the Caldirola–Kanai oscillator. The eigenvalue, $\alpha(t)$, of the lowering operator is given by Eq. (68) whose magnitudes are constant with time. We confirmed that the probability density of the even and odd coherent states and the Perelomov coherent states for the Caldirola–Kanai oscillator converges



(a)



(b)

Fig. 4. Same as in Fig. 3, but with a driving force Eq. (57). We used $f_0 = 20$, $\omega_d = 4.5$, and $\phi_d = 0$.

to the center as time goes by, due to the damping constant γ. All the figures of probability density for the system with a driving force are somewhat deformed.

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